Indian Statistical Institute
Backpaper Examination
Algebra I
2017-2018

Max Marks: 100 Time: 3 hours.

Answer all questions.

1. Give examples of the following.
(a) subgroups K, H of a group G such that $K \trianglelefteq H$, $H \trianglelefteq G$ but K is not a normal subgroup of G.
(b) a group G such that the inner automorphism group $\mathrm{Inn}(G)$ is isomorphic to G.
(c) a group G such that $G \cong G \times G$.
(d) a subgroup H of a group G which is normal but not characteristic.
(e) a group which cannot be written as a direct product of two of its proper subgroups. (6 × 5)

2. (a) State and prove the *class equation*.
(b) Prove that if p is a prime and G is a group of prime power order p^n for some $n \geq 1$, then G has nontrivial center, ie. $Z(G) \neq \{e\}$. (10+5)

3. (a) Show that two elements in S_n belong to the same conjugacy class if and only if they have the same cycle type.
(b) Find the number of elements in the centraliser $C_{S_n}(\sigma)$ of an m-cycle σ in S_n for $m \leq n$. Write down the elements explicitly. (10+5)

4. Let G be a group and H a subgroup of G. Prove that if H has finite index n then there is a normal subgroup K of G with $K \subseteq H$ and $|G : K| \leq n!$. (10)

5. (a) Define commutator subgroup of a group G. Show that the commutator subgroup is a normal subgroup of G.
(b) Show that if H is a subgroup of G, and H intersects the commutator subgroup of G trivially, then $H \subseteq Z(G)$. (10+5)

6. (a) Define (external) semidirect product of two groups H and K.
(b) Classify all groups of order pq, where p, q are primes. (5+10)
